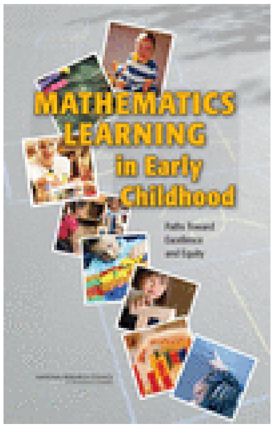

Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity

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2

Foundational Mathematics Content

Mathematics provides a powerful means for understanding and analyzing the world. Mathematical ways of describing and representing quantities, shapes, space, and patterns help to organize people's insights and ideas about the world in systematic ways. Some of these mathematical systems have become such a fundamental part of people's everyday lives—for example, counting systems and methods of measurement—that they may not recognize the complexity of the ideas underpinning them. In fact, the mathematical ideas that are suitable for preschool and the early grades reveal a surprising intricacy and complexity when they are examined in depth. At the deepest levels, they form the foundations of mathematics that have been studied extensively by mathematicians over centuries (e.g., see Grattan-Guinness, 2000) and remain a current research topic in mathematics.

In this chapter, we provide an overview of the mathematical ideas that are appropriate for preschool and the early grades and discuss some of the more complex mathematical ideas that build on them. These foundational ideas are taken for granted by many adults and are not typically examined in high school or college mathematics classes. Thus, many people with an interest in early childhood education may not have had adequate opportunities in their preparation to examine these ideas. Chapters 5 and 6 examine these ideas again in some detail, from the perspective of how children come to understand them and the conceptual connections they make in doing so.

This chapter has four sections. The first two describe mathematics for young children in two core areas: (1) number and (2) geometry and measurement. These ideas, which are important preparation for school and for life, are also genuinely mathematical, with importance from a mathematician's perspective. Moreover, they are interesting to children, who enjoy engaging with these ideas and exploring them.

The third section describes mathematical process goals, both general and specific. The general process goals are used throughout mathematics, in all areas and at every level, including in the mathematics for very young children. The specific process goals are common to many topics in mathematics. These process goals must be kept in mind when considering the teaching and learning of mathematics with young children.

The fourth section describes connections across the content described in the first two sections as well as to important mathematics that children study later in elementary school. These connections help to demonstrate the foundational nature of the mathematics described in the first two sections.

NUMBER CONTENT

Number is a fundamental way of describing the world. Numbers are abstractions that apply to a broad range of real and imagined situations—five children, five on a die, five pieces of candy, five fingers, five years, five inches, five ideas. Because they are abstract, numbers are incredibly versatile ways of explaining the world. "Yet, in order to communicate about numbers, people need representations—something physical, spoken, or written" (National Research Council, 2001, p. 72). Understanding number and related concepts includes understanding concepts of quantity and relative quantity, facility with counting, and the ability to carry out simple operations. We group these major concepts into three core areas: number, relations, and operations. Box 2-1 summarizes the major ideas in each core area. Developing an understanding of number, operations, and how to represent them is one of the major mathematical tasks for children during the early childhood years.

The Number Core

The number core concerns the list of counting numbers 1, 2, 3, 4, 5, . . . and its use in describing how many things are in collections. There are two distinctly different ways of thinking about the counting numbers: on one hand, they form an ordered list, and, on the other hand, they describe cardinality, that is, how many things are in a set. The notion of 1-to-1 correspondence bridges these two views of the counting numbers and is also central to the notion of cardinality itself. Another subtle and important aspect of numbers is the way one writes (and says) them using the base 10

BOX 2-1
Overview of Number, Relations, and Operations Core

The Number Core: Perceive, Say, Describe/Discuss, and Construct Numbers

Cardinality: giving a number word for the numerosity of a set obtained by perceptual subitizing (immediate recognition of 1 through 3) or conceptual subitizing (using a number composition/decomposition for larger numerosities), counting, or matching.

Number word list: knowing how to say the sequence of number words.

1-to-1 counting correspondences: counting objects by making the 1-to-1 time and spatial correspondences that connect a number word said in time to an object located in space.

Written number symbols: reading, writing, and understanding written number symbols (1, 2, 3, etc.).

Coordinations across the above, such as using the number word list in counting and counting to find the cardinality of a set.

The Relations Core: Perceive, Say, Describe/Discuss, and Construct the Relations More Than, Less Than, and Equal To on Two Sets by

Using general perceptual, length, density strategies to find which set is more than, less than, or equal to another set, and then later.

Using the unitizing count and match strategies to find which set is more than, less than, or equal to another set, and then later.

Seeing the difference between the two sets, so the relational situation becomes the additive comparison situation listed below.

The Operations Core: Perceive, Say, Describe/Discuss, and Construct the Different Addition and Subtraction Operations (Compositions/Decompositions of Numbers)

Change situations: addition change plus situations (start + change gives the result) and subtraction change minus situations (start – change gives the result).

Put together/take apart situations: put together two sets to make a total; take apart a number to make two addends.

Compose/decompose numbers: Move back and forth between the total and its composing addends: "I see 3. I see 2 and 1 make 3."

Embedded number triads: Experience a total and addends hiding inside it as a related triad in which the addends are embedded within the total.

Additive comparison situations: Comparing two quantities to find out how much more or how much less one is than the other (the Relations Core precedes this situation).

system. The top section of Box 2-1 provides an overview of the number core from the perspective of children's learning; this is discussed in more detail in Chapter 5. Here we discuss the number core from a mathematical perspective, as a foundation for the discussion of children's learning.

Numbers Quantify: They Describe Cardinality

Numbers tell "how many" or "how much." In other words, numbers communicate how many things there are or how much of something there is. One can use numbers to give specific, detailed information about collections of things and about quantities of stuff. Initially, some toy bears in a basket may just look like "some bears," but if one knows there are seven bears in the basket, one has more detailed, precise information about the collection of bears.

Numbers themselves are an abstraction of the notion of quantity because any given number quantifies an endless variety of situations. We use the number 3 to describe the quantity of three ducks, three toy dinosaurs, three people, three beats of a drum, and so on. We can think of the number 3 as an abstract, common aspect that all these limitless examples of sets of three things share.

How can one grasp this common aspect that all sets of three things share? At the heart of this commonality is the notion of 1-to-1 correspondence. Any two collections of three things can be put into 1-to-1 correspondence with each other. This means that the members of the first collection can be paired with the members of the second collection in such a way that each member of the first collection is paired with exactly one member of the second collection, and each member of the second collection is paired with exactly one member of the first collection. For example, each duck in a set of three ducks can be paired with a single egg from a set of three eggs so that no two ducks are paired with the same egg, no two eggs are paired with the same duck, and no ducks or eggs remain unpaired.

The Number List

The counting numbers can be viewed as an infinitely long and ordered list of distinct numbers. The list of counting numbers starts with 1, and every number in the list has a unique successor. This creates a specific order to the counting numbers, namely 1, 2, 3, 4, 5, 6, ... It would not be correct to leave a number out of the list, nor would it be correct to switch the order in which the list occurs. Also, every number in the list of counting numbers appears only once, so it would be wrong to repeat any of the numbers in the list.

The number list is useful because it can be used as part of 1-to-1 object counting to tell how many objects are in a collection. Although the number of objects in small collections (up to 3 or 4) can be recognized immediately—this is called *subitizing*—in general, one uses the number list to determine the number of objects in a set by counting. Counting allows one to quantify exactly collections that are larger than can be immediately recognized. To count means to list the counting numbers in order, usually starting at 1, but sometimes starting at another number, as in 5, 6, 7, ... (Other forms of counting include "skip counting," in which one counts every second, or third, or fourth, etc., number, such as 2, 4, 6, . . . , and counting backward, as in 10, 9, 8, 7, ...)

Although adults take it for granted because it is so familiar, the connection between the list of counting numbers and the number of items in a set is deep and subtle. It is a key connection that children must make. There are also subtleties and deep ideas involved in saying and writing the number list, which adults also take for granted because their use is so common. Because of the depth and subtlety of ideas involved in the number list and its connection to cardinality, and because these ideas are central to all of mathematics, it is essential that children become fluent with the number list (see Box 2-2).

Connecting the number list with cardinality. In essence, counting is a way to make a 1-to-1 correspondence between each object (in which the

BOX 2-2
The Importance of Fluency with the Number List

All of the work on the relations/operation core in kindergarten serves a double purpose. It helps children solve larger problems and become more fluent in their Level 1 solution methods. It also helps them reach fluency with the number word list in addition and subtraction situations, so that the number word list can become a representational tool for use in the Level 2 counting of solution methods. To get some sense of this process, try to add or subtract using the alphabet list instead of the number word sequence. For counting on, you must start counting with the first addend and then keep track of how many words are counted on. Many adults cannot start counting within the alphabet from D or from J because they are not fluent with this list. Nor do they know their fingers as letters (How many fingers make F?), so they cannot solve D + F by saying D and then raising a finger for each letter said after D until they have raised F fingers. It is these prerequisites for counting on that kindergarten children are learning as they count, add, and subtract many, many times. Of course as they do this, they will also begin to remember certain sums and differences as composed/decomposed triads (as *number facts*).

objects can be any discrete thing, from a doll, to a drumbeat, to the idea of a unicorn) and a prototypical set, namely a set of number words. For example, when a child counts a set of seven bears, the child makes a 1-to-1 correspondence between the list 1, 2, 3, 4, 5, 6, 7 and the collection of bears. To count the bears, the child says the number word list 1, 2, 3, 4, 5, 6, 7 while pointing to one new bear for each number. As a result, each bear is paired with one number, each number is paired with one bear, and there are no unpaired numbers or bears once counting is completed. The pairing could be carried out in many different ways (starting with any one of the bears and proceeding to any other bear next, and so on), but any single way of making such a 1-to-1 correspondence by counting establishes that there are seven bears in the set.

A key characteristic of object counting is that the last number word has a special status, as it specifies the total number of items in a collection. For example, when a child counts a set of seven bears, the child counts 1, 2, 3, 4, 5, 6, 7, pointing to one bear for each number. The last number that is said, 7, is not just the last number in the list, but also indicates that there are seven bears in the set (i.e., cardinality of the set). Thus when counting the 7 bears, the counter shifts from a counting reference (to 7 as the last bear when counting) to a cardinal reference when referring to 7 as the number of bears in all. Counting therefore provides another way to grasp the abstract idea that all sets of a fixed number of things share a common characteristic—that when one counts two sets that have the same number of objects, the last counting word said will be the same for both.

Another key observation about counting is that, for any given number in the list of counting numbers, the next number in the list tells how many objects are in a set that has one more object than do sets of the given number of objects. For example, if there are five stickers in a box and one more sticker is put into the box, then one knows even without counting them all again that there will now be six stickers in the box, because 6 is the next number in the counting list. Generally each successive counting number describes a quantity that is one more than the quantity that the previous number describes.

In a sense, then, counting is adding: Each counting number adds one more to the previous collection (see Figure 2-1). Of course, if one counts backward, then one is subtracting. These observations are essential for children's early methods of solving addition and subtraction problems. Also, each step in the counting process can be thought of as describing the total number of objects that have been counted so far.

The number word list and written number symbols in the base 10 place-value system. Each number in the number list has a unique spoken name and can be represented by a unique written symbol. The names and symbols for the initial numbers in the list have been passed along by tradition, but

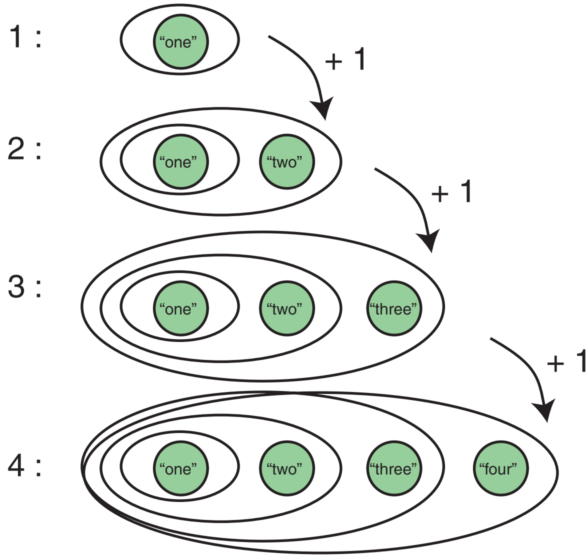

FIGURE 2-1 Each counting number describes a quantity that is one more than the previous number describes.

the English names of the first 10 (or so) counting numbers and the symbols of the first 9 counting numbers are arbitrary and could have been different. For example, instead of the English word "three," one could be using "bik" or "Russell" or any other word, such as the words for "three" in other languages. Instead of the symbol 3, one could use a symbol that looks completely different.

The list of counting numbers needs to go on and on in order to count ever larger sets. So the problem is how to give a unique name to each number. Different cultures have adopted many different solutions to this problem (e.g., Menninger, 1958/1969; see Chapter 4 of this volume for a discussion of counting words in different languages). The present very efficient solution to this problem was not obvious and was in fact a significant achievement in the history of human thought (Menninger, 1958/1969). Even though the first nine counting numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, are represented by distinct, unrelated symbols, some mechanism for continuing to list numbers without resorting to creating new symbols indefinitely is desirable.

The decimal system (or base 10 system) is the ingenious system used today to write (and say) counting numbers. The decimal system allows one to use only the 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to write any counting number as a string of digits (such a written representation of a number is often called a *numeral*).

The system is called a base 10 system because it uses 10 distinct digits and is based on repeated groupings by 10. The use of only 10 digits to write any counting number, no matter how large, is achieved by using *place value*. That is, the meaning of a digit in a written number depends (in a very specific way) on its placement. The details about using the decimal system

BOX 2-3
Using the Decimal System to Write the List of Counting Numbers

Each of the first nine counting numbers (or number words) "one, two, . . . , nine," requires only one digit to write, 1, 2, . . . , 9. Each digit stands for that many things—in other words, that many "ones," as indicated at the top of Figure 2-2. Each of these digits is viewed as being in the "ones place."

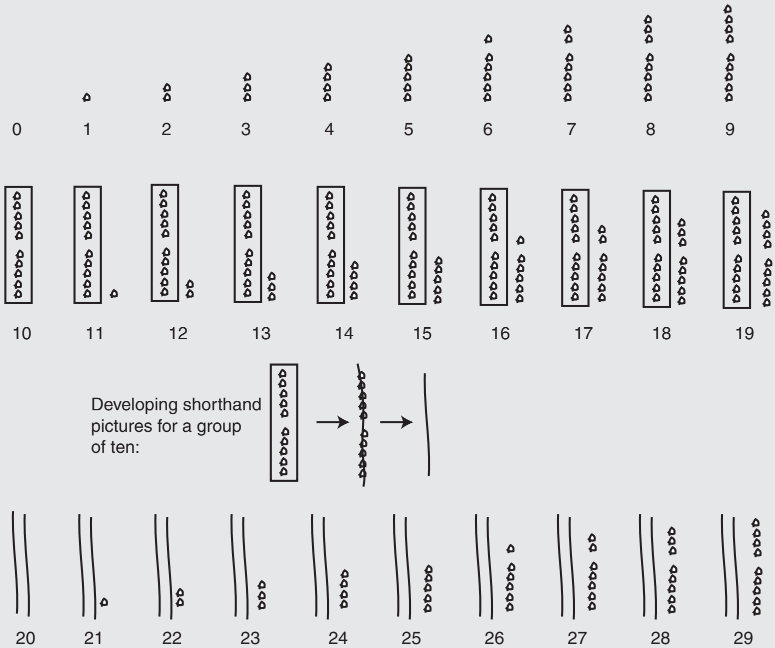

FIGURE 2-2 Decimal system 1.

The next counting number, ten, requires two digits to write. The 1 stands for 1 ten and the 0 stands for 0 ones, and 10 stands for the combined amount in 1 ten and 0 ones. This way of describing and writing the number ten requires thinking of it as a single group of ten—in other words, as a new entity in its own right, which is created by joining 10 separate things into a new coherent whole, as indicated in the figure by the way 10 dots are shown grouped to form a single unit of 10.

to write the list of counting numbers are given in Box 2-3: A key idea is to create larger and larger units, which are the values of places farther and farther to the left, by taking the value of each place to be 10 times the value of the previous place to its right. One can think of doing this by bundling together 10 of the previous place's value. The greater and greater values

In each of the next two-digit counting numbers, 11, 12, 13, 14, 15, . . . , 20, 21, 22, . . . , 30, 31, . . . , 97, 98, 99, the digit on the right stands for that many ones, so one says this digit is in the "ones place," and the digit on the left stands for that many tens, so one says it is in the "tens place"; the number stands for the combined amount in those tens and ones. For example, in 37, the 3 stands for 3 tens, the 7 stands for 7 ones, and 37 stands for the combined amount in 3 tens and 7 ones. Notice that from 20 on, the way one says number words follows a regular pattern that fits with the way these numbers are written. But the way one says 11 through 19 does not fit this pattern. In fact, 13 through 19 are said backward, because the ones digit is said before the tens digit is indicated.

The number 99 is the last two-digit counting number, and it stands for the combined amount in 9 tens and 9 ones (see Figure 2-3). The next counting number will be the number of dots there are when one more dot is added to the dots on the left of the figure. This additional dot "fills up" a group of ten, as indicated in the middle of the figure. Now there are 10 tens, but there isn't a digit that can show this many tens in the tens place. So the 10 tens are bundled together to make a new coherent whole, as indicated on the right in Figure 2-3, which is called a hundred. From 0 to 9 hundreds can be recorded in the place to the left of the tens place, which is called the hundreds place. So the next counting number after 99 is written as 100, in which the 1 stands for 1 hundred, and the 0s stand for 0 tens and 0 ones.

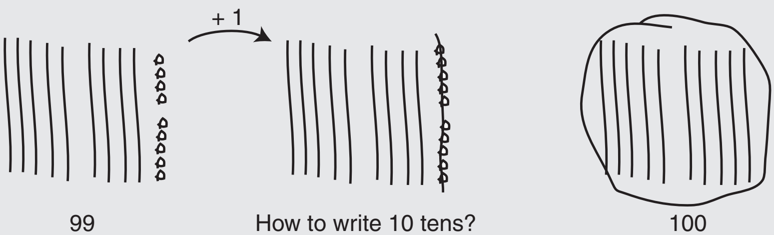

FIGURE 2-3 Decimal system 2.

The decimal system has a systematic way to make new larger units by bundling 10 previously made units and recording the new unit one place to the left of the given unit's place. Just as 10 ones make a new unit of 10, which is recorded to the left of the ones place, 10 tens make a new unit of a hundred, which is recorded to the left of the tens place, and 10 hundreds make a new unit of a thousand, which is recorded to the left of the hundreds place. This pattern continues on and on to new places on the left.

of the places allow any number, no matter how large, to be expressed as a combination of between 0 and 9 of each place's value. In this way, every counting number can be expressed in a unique way as a numeral made of a string of digits. (See Howe, 2008, for a further discussion of the decimal system and place value.)

Even though most countries around the world now use this system of written numerals, they still use their own list of counting words that relate closely, or not so closely, to the written system of numerals. English and other European lists of counting words have various aspects that do not fit the decimal system so well and that create difficulties in learning the system. These, and ways to compensate for these difficulties, are discussed in Chapter 4.

The Relations/Operations Core

Numbers do not exist in isolation. They make up a coherent system in which numbers can be compared, added, subtracted, multiplied, and divided. Just as numbers are abstractions of the notion of quantity, the relations "less than," "greater than," and "equal to" and the operations of addition, subtraction, multiplication, and division are abstractions of comparing, combining, and separating quantities. These relations and operations apply to a wide variety of problems. The middle and bottom sections of Box 2-1 are an overview of the relations core and the operations core for young children (which concerns only addition and subtraction, not multiplication or division).

Comparing

In some cases it is visually evident that there are more things in one collection than in another, such as in the case of the two sets of beads shown at the top of Figure 2-4. But in other cases it is not immediately clear which collection (if either) has more items in it.

A basic way to compare two collections of objects is by direct matching (as in the middle of Figure 2-4). If a child has a collection of black beads and another collection of white beads, and if these collections are placed near each other, the child can place each black bead with one and only one white bead. If there is at least one extra white, then there are more whites; if at least one extra black, then more blacks. And if none is left over, then the two groups have the same number (although one may not know and does not need to know exactly what number it is).

When direct matching is not possible, a child can count the number of beads in two collections to determine which collection (if either) has more beads or if they both have the same number of beads. A key observation

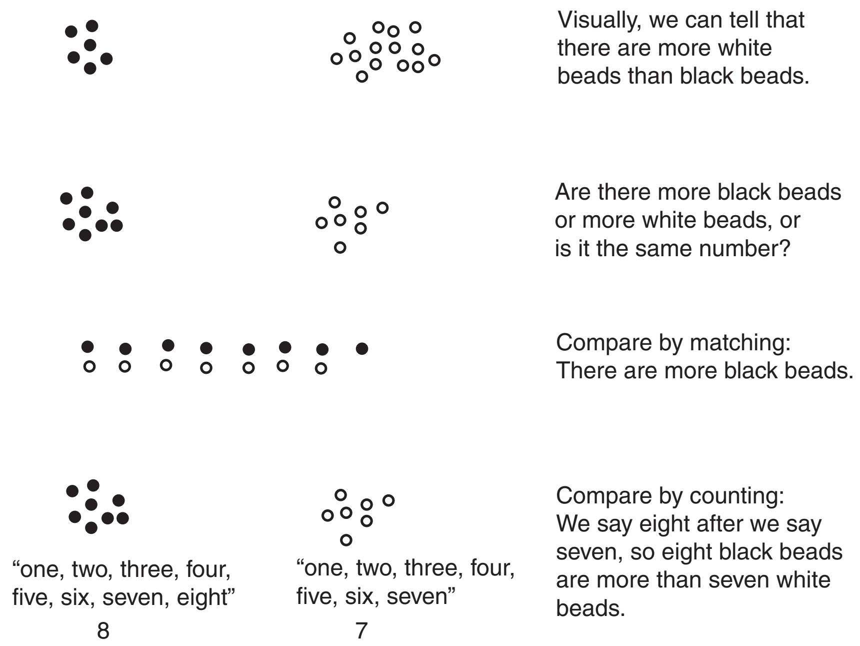

FIGURE 2-4 Comparing.

about using counting to compare is that a number that is said later in the counting word list corresponds to a collection that has a greater number of objects than does a collection corresponding to a number earlier in the sequence. For example, one knows that there are more beads in a collection of eight black beads than there are in a collection of seven white beads because 8 occurs later in the counting list than 7 (see the bottom of Figure 2-4). Counting thus provides a more advanced way to compare sets of things than direct matching because it relies on knowledge about how numbers compare. Counting is also a more powerful way to compare sets of things than direct matching because it allows sets that are not in close proximity to be compared.

A key point about comparing collections of objects is that counting can be used to do so, and it relies on the link between the number list and cardinality: Numbers later in the list describe greater cardinalities than do numbers earlier in the list. Finding out which collection is more than another collection is easier than determining exactly how many more that collection has than the other, which can be formulated as an addition or subtraction problem. This more specific version of comparison is discussed in the next section.

Addition and Subtraction Story Problems and Situations

Addition and subtraction are used to relate amounts before and after combining or taking away, to relate amounts in parts and totals, or to say precisely how two amounts compare. Story problems and situations that can be formulated with addition or subtraction occur in a wider variety than just the simplest and most common "add to" and "take away" story problems. Methods that young children can use to solve addition and subtraction story problems, again, rely on a fluent link between the number list and cardinality. Later methods (in first grade or so) also rely on decomposing numbers and on an initial understanding of the base 10 system, namely that the numbers 11 through 19 can be viewed as a ten and some ones.

Box 2-4 describes the different types of story problems or situations that can be formulated with addition or subtraction. Viewed from a more

BOX 2-4
Types of Addition/Subtraction Situations

Change Plus and Change Minus Situations

Change situations have three quantitative steps over time: start, change, result. Most children before first grade solve only problems in which the result is the unknown quantity. In first grade, any quantity can be the unknown number. Unknown start problems are more difficult than unknown change problems, which are more difficult than unknown result problems.

Change plus: Start quantity + change quantity = result quantity: "Two bunnies sat on the grass. One more bunny hopped there. How many bunnies are on the grass now?"

Change minus: Start quantity – change quantity = result quantity: "Four apples were on the table. I ate two apples. How many apples are on the table now?"

Put Together/Take Apart Situations

In these situations, the action is often conceptual instead of physical and may involve a collective term like "animal": "Jimmy has one horse and two dogs. How many animals does he have?"

In put together situations, two quantities are put together to make a third quantity: "Two red apples and one green apple were on the table. How many apples are on the table?"

In take apart situations, a total quantity is taken apart to make two quantities: "Grandma has three flowers. How many can she put in her red vase and how many in her blue vase?"

These situations are decomposing/composing number situations in which children shift from thinking of the total to thinking of the addends. Working with differ-advanced perspective, most of these situations can be formulated in a natural way with an equation of the form

$$A + B = C \quad \text{or} \quad A - B = C$$

in which two of the three numbers in the equation are known and the problem is to determine the other number that makes the equation true. The types of situations that are naturally formulated with these equations are *change plus* and *change minus* situations, *put together* situations, and *comparison* situations.

In change plus and change minus situations, there is a starting quantity (A), an amount by which this quantity changes (B), and the resulting quantity (C). Problems in which A and B are the known amounts and C is to be determined are the classic, most readily recognized addition and

ent numbers helps them learn number triads related by this total-addend-addend relationship, which they can use when adding and subtracting. Eventually with much experience, children move to thinking of embedded number situations in which one considers the total and the two addends (partners) that are "hiding inside" the total simultaneously instead of needing to shift back and forth.

Equations with the total alone on the left describe take apart situations: $3 = 2 + 1$. Such equations help children understand that the = sign does not always mean *makes* or *results in* but can also mean *is the same number as*. This helps with algebra later.

Comparison Situations

Children first learn the comparing relations equal to, more than, and less than for two groups of things or two numbers. They find out which one is bigger and which one is smaller (or if they are equal) by matching and by counting.

Eventually first grade children come to see the third quantity involved in a more than/less than situation: the amount more or less (the difference). Children then can solve additive comparison problems in which a larger quantity is compared to a smaller quantity to find the difference. Children may write different equations to show such comparisons and may also still solve by matching or counting. As with the other addition and subtraction situations, any of the three quantities can be unknown. The language involved in such situations is complex because the comparing sentence gives two kinds of information. "Julie has six more than Lucy" says both that "Julie has more than Lucy" and that the amount more is six. This is a difficult linguistic structure for children to understand and to say.

NOTE: Researchers use different names for these types of addition and subtraction situations, and some finer distinctions can be made within the categories. However, there is widespread agreement about the basic types of problem situations despite the use of different terminology.

subtraction problems. Reversing the action in change minus or change plus situations shows the connection between subtraction and addition. For example, if Whitney had 9 dinosaurs and gave away 3 dinosaurs, how many dinosaurs did Whitney have left? This problem can be formulated with the subtraction equation, 9 – 3 = ? Starting with the dinosaurs Whitney has left, if she gets the 3 dinosaurs back, she will have her original 9 dinosaurs, which can be expressed with the addition equation ? + 3 = 9. Subtraction problems can thus be reformulated in terms of addition, which connects subtraction to addition.

In put together situations, there are two parts, A and B, which together make a whole amount, C. These situations are formulated in a natural way with an addition equation, A + B = C.

Change plus, change minus, and put together problems in which either A or B (the start quantity, the change quantity, or one of the two parts) is unknown involve an interesting reversal between the operation that formulates the problem and the operation that can be used to solve the problem from a more advanced perspective. For example, consider this "change unknown" problem: "Matt had 5 cards. After he got some more cards, he had 8. How many cards did Matt get?" This problem can be formulated with the addition equation 5 + ? = 8. Although young children will solve this problem by adding on to 5 until they reach 8 (perhaps with actual cards or other objects), older children and adults may solve the problem by subtracting, 8 – 5 = 3, which uses the opposite operation than the addition equation that was used to formulate the problem.

Comparison situations concern precise comparisons between two different quantities, A and C. Instead of simply saying that A is greater than, less than, or equal to C, the situation concerns the exact amount by which the two quantities differ. If C is B more than A, then the situation can be formulated with the equation A + B = C. If C is B less than A, then the situation can be formulated with the equation A – B = C. To consider this precise difference, B, requires one to conceptually create a collection that is not physically present separately in the situation. This difference is either that part of the larger collection that does not match the smaller collection, or it is those objects that must be added to the smaller collection to match the larger collection. Of course, these matches can be done by counting and with specific numbers rather than just by matching. Note that these situations are called additive comparison situations even when formulated with subtraction (A – B = C when C is B less than A) to distinguish them from multiplicative comparison situations, which can be formulated in terms of multiplication or division. Students solve multiplicative comparison problems in the middle and later elementary grades.

In take apart situations, a total amount, C, is known and the problem is to find the ways to break the amount into two parts (which do not have to be equal). Take apart situations are most naturally formulated with an equation of the form

$$C = A + B$$

in which C is known and all the possible combinations of A and B that make the equation true are to be found. There are usually many different As and Bs that make the equation true.

GEOMETRY/MEASUREMENT CONTENT

Geometry and measurement provide additional, powerful systems for describing, representing, and understanding the world. Both support many human endeavors, including science, engineering, art, and architecture. Geometry is the study of shapes and space, including two-dimensional (2-D) and three-dimensional (3-D) space. Measurement is about determining the size of shapes, objects, regions, quantities of stuff, or quantifying other attributes. Through their study of geometry and measurement, children can begin to develop ways to mentally structure the spaces and objects around them. In addition, these provide a context for children to further develop their ability to reason mathematically.

Every 3-D object or 2-D shape, even very simple ones, has multiple aspects that can be attended to: the overall shape, the particular parts and features of the object or shape, and the relationships among these parts and with the whole object or shape. In determining the size of a shape or object, one must first decide on which particular aspect or measurable attribute to focus.

Space (both 3-D and 2-D) could be viewed initially as an empty, unstructured whole, but objects that are placed or moved within the space begin to structure it. The beginnings of the Cartesian structure of space, a central idea in mathematics, are seen when square tiles are placed in neat arrays to form larger rectangles and when cubical blocks are stacked and layered to make larger box-shaped structures. These are also examples of composing and decomposing shapes and objects more generally. Composing and decomposing shapes and objects are part of a foundation for later reasoning about fractions and about area and volume.

Viewing or imagining an object from different perspectives in space and moving or imagining how to move an object through space to fit in a particular spot links spatial relations with the parts and features of objects and shapes.

Just as numbers are an abstraction of quantity, the ideal, theoretical shapes (2-D and 3-D) of geometry are an abstraction of their approximate physical versions. The angles in a rectangular piece of paper aren't exactly right angles, the edges aren't perfectly straight line segments, and the paper, no matter how thin, has a thickness to it that makes it a solid 3-D shape rather than only 2-D. Measurements of actual physical objects are never exact, either. Even so, valid reasoning about ideal geometric shapes and ideal theoretical measurements can be aided with approximate physical shapes and measurements.

Measurement

In its most basic form, measurement is the process of determining the size of an object. But the size of an object can be described in different ways, depending on the attribute one chooses. For example, the size of a tower made of cube-shaped blocks might be described by the height of the tower (a length) or in terms of the number of blocks in the tower (a volume). The size of the floor of a room that is covered in square tiles can be described in terms of the number of tiles on the floor (an area). The most important measurable attributes in mathematics are length, area, and volume.

To measure a quantity (with respect to a given measurable attribute, such as length, area, or volume), a unit must be chosen. Once a unit is chosen, the size of an object (with respect to the given measurable attribute) is the number of those units it takes to make (the chosen attribute of) the object.

For length, a stick, for example, 1 foot long, could be chosen to be a unit. With respect to that unit of length, the length of a toy train is the number of those sticks (all identical) needed to lay end to end alongside the train from the front to the end.

For area, a square tile, such as a tile that is 1 inch by 1 inch, could be chosen to be a unit. With respect to that unit of area, the area of a rectangular tray is the number of those tiles (all identical) it takes to cover the tray without gaps or overlaps. Although squares need not be used for units of area, they make especially useful units because they line up in neat rows and columns and fill rectangular regions completely without gaps or overlaps.

For volume, a cube-shaped block, such as a block that is 1 inch by 1 inch by 1 inch, could be chosen to be a unit. With respect to that unit of volume, the volume of a box is the number of those cubes (all identical) it takes to fill the box without any gaps. Although cubes need not be used for units of volume, they make especially useful units because they line up in neat rows and columns and stack in neat layers to fill box shapes completely without gaps.

Once a unit has been chosen, a measurement is a number of those units (e.g., 3 inches, 6 square inches, 12 cubic inches). So measurement is a generalization of cardinality, which describes how many things are in a collection. For young children, measurements will generally be restricted to whole numbers, but measurement is a natural context in which fractions arise. To fill a bucket with sand, a child might pour in 4 full cups of sand and another cup that is only half full of sand, so that the volume of the bucket is approximately 4½ cups.

An important but subtle idea about units, which children learn gradually, is that when measuring a given object, the larger the unit used to measure, the smaller the total number of units. For example, suppose there are two sizes of sticks to use as units of length: short sticks and longer sticks. More short sticks than long ones are needed to measure the same length. In other words, there is an inverse relation between the size of a measuring unit and the number of units needed to measure some characteristic.

Young children may also not grasp the importance of using standard units, which allow one to compare objects that are widely separated in space or time (see Chapter 3 for further discussion).

2-D Shapes

Shapes found in nature, such as flowers, leaves, tree trunks, and rocks, are complex, intricate, and 3-D rather than 2-D. In contrast, the familiar 2-D shapes studied in geometry, such as triangles, rectangles, and circles, are relatively simple. Compared with most shapes in the natural world, these shapes are relatively easy to draw or create and also to describe and analyze. Many manufactured objects, such as tabletops and appliances, have parts that are approximate triangles, rectangles, or circles. Many shapes in the natural world are approximate combinations of parts of these simpler geometric shapes. For example, a birch leaf might look like a triangle joined to a half-circle.

Although geometric shapes can be described and discussed informally and children can simply be told the names of some prototypical examples of these shapes (for ease of reference and discussion), these shapes also have mathematical definitions, which teachers should know.

Parts and Features of 2-D Shapes

Geometric shapes have parts and features that can be observed and analyzed. The shapes all have an "inside region" and an "outer boundary." Distinguishing the inside region of a 2-D shape from its outer boundary is an especially important foundation for understanding the distinction between the perimeter and area of a shape in later grades. Except for circles, the outer boundary of the common 2-D geometric shapes consists of straight sides, and the nature of these sides and their relationships to each other are important characteristics of a shape. One can attend to the number of sides and the relative length of the sides: Are all the sides of the same length, or are some longer than others? Where two sides meet, there is a corner point or vertex (plural: vertices). One interesting observation is that the number of vertices is the same as the number of sides. One can attend to how "pointy" a shape is at its vertex. In this case, one is attending to the angle formed by the sides that meet at the vertex. In some shapes, all the angles are the same, such as rectangles. In some shapes, some angles are the same and others are different, such as a rhombus that is not a square. The study of geometry is not only about seeing shapes as wholes; it's about finding and analyzing their properties and features.

Additional Characteristics of 2-D Shapes Beyond Their Defining Characteristics

In studying shapes, young children's attention will be drawn to the many different characteristics and features of a given shape. But from a more advanced standpoint, mathematicians have made definitions of shapes precise and spare by selecting only some of the characteristics of a shape as defining characteristics. For example, the definition of a triangle is a 2-D shape with three straight sides. A triangle also has three vertices and three angles, but these are not mentioned in the definition of triangle. Similarly, the opposite sides in a rectangle are the same length, but this is not mentioned in the definition of rectangle. Young children, however, can observe and describe these additional properties of shapes. For example, when one folds a rectangle out of paper by folding right angles, one can see that the opposite sides of the rectangle are the same length. The rectangle wasn't constructed with the explicit intent of making opposite sides the same length, yet it turns out that way. Similarly, if one joins four sticks end to end to make a quadrilateral and if the sticks were chosen so that the opposite sides are the same length, one can see that the opposite angles are also the same. Although the shape wasn't constructed with the explicit intent of making opposite angles the same, it nevertheless turns out that way.

3-D Shapes

The common simple geometric 3-D shapes are cubes, prisms, cylinders, pyramids, cones, and spheres. Many common objects are approximate versions of these ideal, theoretical shapes. For example, a building block is a rectangular prism, and a party hat can be in the shape of a cone. As with 2-D shapes, the study of 3-D shapes is not only about seeing these shapes as wholes and learning their names, but also about finding and analyzing their properties and features.

The 3-D geometric shapes have parts and features that can be observed. The shapes all have an "inside" and an "outer surface." The outer surface may consist of several parts. For example, the outer surface of a prism can consist of rectangles. If the outer surface of a 3-D shape consists of flat surfaces, these are often called faces. For example, a long wooden building block has two faces at each end that are small rectangles and four faces around the middle that are long rectangles. Faces are joined along straight edges, and edges meet at points called vertices. Children might observe that some shapes (like that building block) have pairs of faces on opposite sides that are the same (congruent). Children might also observe that some shapes, like cylinders (like a pole or a can), cones (like a party hat), and spheres (like a ball), have outer surfaces that are not flat.

Although the outer surface of a 3-D shape is usually visible, unless one cuts the shape open, or the shape is made of clear plastic, or the shape is hollow and a face can be removed to look inside, one must usually imagine and visualize the inside. One exception is rooms, which are often (roughly) in the shape of a rectangular prism, and which one experiences from the inside. Distinguishing the inside of a 3-D shape from its outer surface is an especially important foundation for understanding the distinction between the surface area and volume of a shape in later grades.

Composing and Decomposing Shapes

Just as 10 ones can be composed to make a single unit of 10, shapes can also be composed to make new, larger shapes. And just as a 10 can be decomposed into 10 ones, so too shapes can be decomposed to make new, smaller shapes. Figure 2-5 presents a few examples of relationships among shapes obtained by composing and decomposing shapes based on equilateral triangles. Figure 2-6 shows relationships among shapes obtained by composing and decomposing rectangles.

Composing and decomposing 2-D shapes is an important foundation for understanding area in later grades. In particular, viewing rectangles as

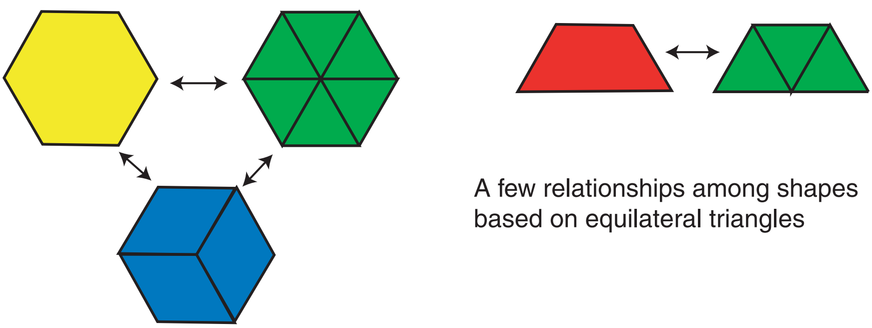

FIGURE 2-5 Relationships among shapes based on equilateral triangles.

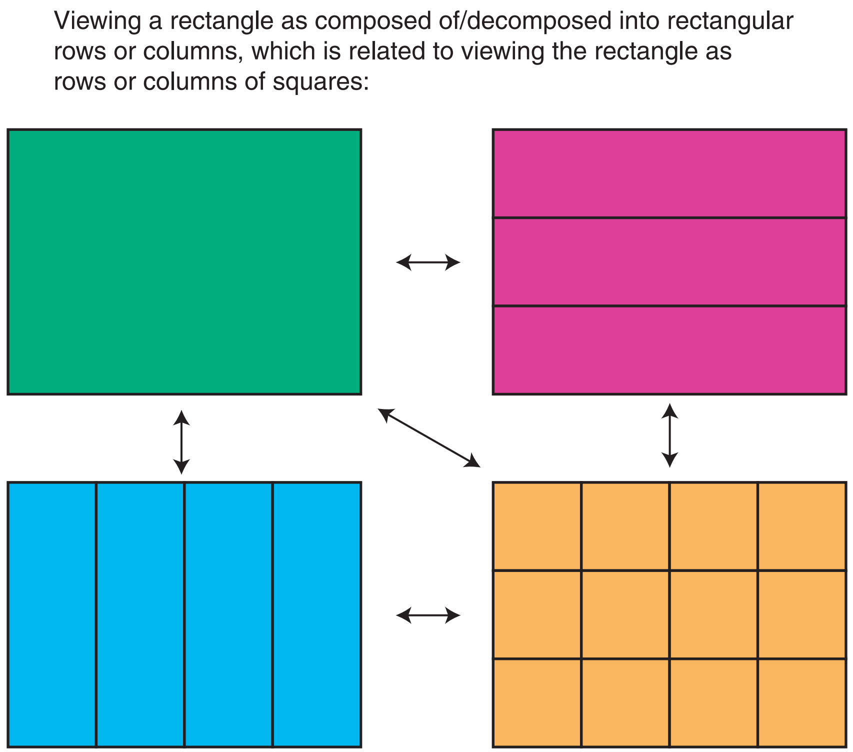

FIGURE 2-6 Relationships among rectangles.

composed of rows and columns of squares, as illustrated in Figure 2-6, is key to understanding areas of rectangles.

Likewise, composing and decomposing 3-D shapes is an important foundation for understanding volume in later grades. In particular, viewing rectangular prisms as composed of layers of rows and columns of cubes is key to understanding volumes of rectangular prisms (see Figure 2-7). Also, reasoning about fractions often takes place in a context of reasoning about decomposing shapes into pieces.

Composition and decomposition is discussed in greater detail in the section on mathematical connections across content areas and to later mathematics.

Motion, Relative Location, and Spatial Structuring

Part of the study of geometry is the analysis of both 2-D and 3-D space. A flat tabletop or piece of paper (imagined to extend infinitely in all

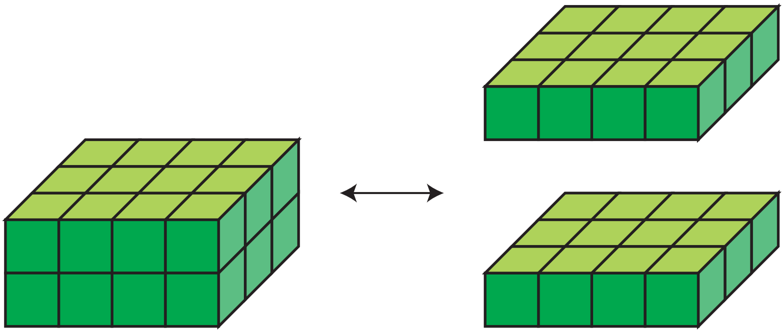

FIGURE 2-7 Viewing a rectangular box as composed of layers of rows and columns.

directions) is a model for 2-D space. The space around one is a model for 3-D space. For young children, the study of space begins with movement through space and with describing relative location in space.

Space is oriented by relative location. Think of one object as at a fixed location in 3-D space. Another object may be above or below the fixed object, which indicates relative location along a vertical axis (line). Another object may be in front of or behind a fixed object, or it may be to the left or right of a fixed object. These two descriptions indicate relative location along two distinct (and perpendicular) horizontal axes (lines). A related way to begin to structure space is to join squares into neat arrays of rows and columns for 2-D space and to stack cubes in layers of rows and columns for 3-D space.

Although objects can be moved through space in many different ways, in 2-D space (think of a 2-D shape on a tabletop) there are some special motions that are of particular interest in advanced geometry that are also accessible to young children. Using elementary school terminology, these motions are called slides, flips, and turns (and in more advanced settings they are called translations, reflections, and rotations).

A slide moves a shape in a single direction for a specified distance without turning the shape. A flip reflects the shape across a line (so that the top and bottom of the shape become reversed). A turn rotates the shape around a fixed point with a specified amount of turning (e.g., a half turn or a quarter turn). (Technically, the center point of rotation need not be the center of the shape or even within the shape, although for young children it will be chosen that way.)

MATHEMATICAL PROCESS GOALS

In addition to coming to understand the specific mathematical concepts discussed so far, children need to develop proficiency in the reasoning processes used in mathematics. In this section we describe two categories of mathematical processes: (1) general mathematical reasoning processes, which are central in every content area and at every level of mathematics, and (2) specific mathematical reasoning processes, which weave through many different content areas. Note that many of the specific reasoning processes were already touched on in the discussions of number, geometry, and measurement. In fact, these specific processes represent powerful, cross-cutting ideas that connect multiple concepts, procedures, or problems and can help children begin to see coherence across topics in mathematics. One major goal of early education should be to stimulate and foster mathematical reasoning.

General Mathematical Reasoning Processes

The National Council of Teachers of Mathematics (NCTM) identified five process standards essential for meaningful and substantive mathematics learning and teaching (National Council of Teachers of Mathematics, 2000): (1) representing (including analyzing representations mathematically and visualizing internally), (2) problem solving, (3) reasoning, (4) connecting, and (5) communicating. These processes are vehicles for children to deepen, extend, elaborate, and refine their thinking and to explore ideas and lines of reasoning. According to NCTM, these processes are to be continually interwoven throughout the teaching and learning of mathematics content—even at the preschool level (see Chapters 5 and 6 for further discussion).

Representing is central in mathematics. Mathematics at every level uses simplified pictures or diagrams to represent a situation and subject it to mathematical analysis. For example, a child hears the story of The Three Bears. She forms a mental image of the three, with the papa bear largest in size, the mama bear next, and then the baby bear. She draws a crude picture of the three, or perhaps uses stick figures, or even lines. All of these are representations—the mental image, the picture, the stick figures, and the lines. The child can use the representations to reason about the objects and to explore ideas about size. Is the mama bear smaller than the papa bear? Is she also bigger than the baby bear? How can she be both bigger and smaller at the same time? Much later, the student can represent this situation as

A > B and B > C and reason that, if this is the case, then A > C. Here the representations are mathematical in the conventional sense. But when used to understand a situation quantitatively or geometrically, images and simple drawings are no less mathematical than are such representations as written numbers or equations, which are universally recognized as mathematical.

According to mathematical educators, "problem solving and reasoning are the heart of mathematics" (National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002, p. 6). In fact, solving problems is both a goal of mathematics learning and a mechanism for doing so. Young children will need support to formulate, struggle with, and solve problems and to reflect on the reasoning they use in doing so. By developing their ability to reason mathematically, children will begin to note patterns or regularities in the world and across the mathematical ideas they are introduced to. They will become increasingly sophisticated in their ability to recognize and analyze the mathematics inherent in the world around them.

Connecting and communicating are particularly important in the preschool years. Children must learn to describe their thinking (reasoning) and the patterns they see, and they must learn to use the language of mathematical objects, situations, and notation. Children's informal mathematical experiences, problem solving, explorations, and language provide bases for understanding and using this formal mathematical language and notation. The informal and formal representations and experiences need to be continually connected in a nurturing "math talk" learning community, which provides opportunities for all children to talk about their mathematical thinking and produce and improve their use of mathematical and ordinary language. Children also need to connect ideas across different domains of mathematics (e.g., geometry and number) and across mathematics and other subjects (e.g., literacy) and aspects of everyday life.

Applying the Process Standards: Mathematizing

Together, the general mathematical processes of reasoning, representing, problem solving, connecting, and communicating are mechanisms by which children can go back and forth between abstract mathematics and real situations in the world around them. In other words, they are a means both for making sense of abstract mathematics and for formulating real situations in mathematical terms—that is, for *mathematizing* the situations they encounter.

The power of mathematics lies in its ability to unify a wide variety of situations and thereby to apply a common problem-solving strategy in seemingly disparate examples. For example, the number 3 applies not only to concrete situations, such as three pencils or three apples, but also to any collection of three things, real or imagined. Thus, the addition problem 3 + 2 = ? provides an abstract formulation for a vast number of actual situations in the world around one. The abstract nature of mathematics is part of its power: Because it is abstract, it can apply to a virtually limitless number of situations. But for children to use this mathematical power requires that they take situations and problems from the world around them and formulate them in mathematical terms. In other words, it requires children to *mathematize* situations.

Mathematizing happens when children can create a model of the situation by using mathematical objects (such as numbers or shapes), mathematical actions (such as counting or transforming shapes), and their structural relationships to solve problems about the situation. For example, children can use blocks to build a model of a castle tower, positioning the blocks to fit with a description of relationships among features of the tower, such as a front door on the first floor, a large room on the second floor, and a lookout tower on top of the roof. Mathematizing often involves representing relationships in a situation so that the relationships can be quantified.

For example, if there are three green toy dinosaurs in one box and five yellow toy dinosaurs in another box, children might pair up green and yellow dinosaurs and then determine that there are two more yellow dinosaurs than green ones because there are two yellow dinosaurs that do not have a green partner. With experience and guidance, children create increasingly abstract representations of the mathematical aspects of the situation. For example, drawing five circles instead of five yellow ducks or drawing a rectangle to represent the side of a box of tissues and, later, writing an equation to model a situation. Children become able to visualize these mathematical attributes mentally, which helps in various kinds of problem solving. Children also need eventually to learn to read and to write formal mathematical notation, such as numerals (1, 2, 6, 10) and other symbols (=, +, –) and to use these symbols in mathematizing situations. Thus, mathematizing involves reinventing, redescribing, reorganizing, quantifying, structuring, abstracting, and generalizing what is first understood on an intuitive and informal level in the context of everyday activity (Clements and Sarama, 2007).

Specific Mathematical Reasoning Processes

Mathematics learning in early childhood requires children to use several specific mathematical reasoning processes, also known as "big ideas," across domains. These big ideas are overarching concepts that connect multiple concepts, procedures, or problems within or across domains or topics and are a particularly important aspect of the process of forming connections. Big ideas "invite students to look beyond surface features of procedures and concepts and see diverse aspects of knowledge as having the same underlying structure" (Baroody, Feil, and Johnson, 2007, p. 26).

Unitizing

Unitizing—finding or creating a mathematical unit—occurs in numerical, geometric, and spatial contexts. When children count, they must create mental units of what they are going to count: single cats, the paws on several cats, or groups of two cats. To measure length, children must select a unit of length measure (for example, they will lay along a length and then count new crayons, feet stepped heel-to-toe along some distance, or inch lengths). To create repeating patterns, children must select and repeat a unit. For example, they might make a bead necklace by repeatedly stringing two cubes then a sphere (their unit). In designing a block building, they might repeatedly place a square, then a triangular block, repeating that unit around the top of their building. When making designs or pictures with pattern blocks, children might join several shapes to make a unit that they repeat throughout the design. To begin to understand the base 10 place-value system, children must be able to view ten ones as forming a single unit of ten. Research suggests a link between being able to view a collection of shapes as a higher order unit and being able to view two-digit numbers as groups of tens and some ones (Clements et al., 1997; Reynolds and Wheatley, 1996). Because the concept of unit underlies core ideas in number and in geometry and measurement, it has been recommended as a central focus for early childhood mathematics education (Sophian, 2007).

Decomposing and Composing

Decomposing and composing are used throughout mathematics at every level and in all topics. In the realm of numbers and operations (addition, subtraction, multiplication, and division), composing and decomposing are used in recognizing the number of objects in a collection, in the meaning of the operations themselves, and in the place-value system. Children can sometimes quickly determine the number of objects in a small collection by viewing the collection as composed of two immediately recognizable collections, such as seeing four counters as composed of a set of three counters and another counter. Composing and decomposing are the basis for the operations of addition and subtraction and later for the operations of multiplication and division. Some key steps toward developing proficiency with arithmetic involve decomposing and composing. Children must be able to decompose numbers from 1 to 10 into all possible pairs and to recognize numbers from 11 to 19 as composed of a ten and some ones. The base 10 place-value system relies on repeated bundling in groups of ten. Proficiency with multidigit addition and subtraction requires being able to compose ten ones as one ten and to decompose one ten as ten ones.

In geometry, shapes can be viewed as composed of other shapes, such as viewing a trapezoid as made from three triangles, or viewing a house shape as made from a triangle placed above a square. Children can compose rows of squares to make rectangles (see Figure 2-6). Many 3-D shapes seen in everyday life can be viewed as composed of shapes that are found in sets of building blocks (or at least approximately so). A juice box might look like a rectangular prism with a (sideways) triangular prism on top. Children can compose layers of cubes to make larger cubes and rectangular prisms.

In measurement, units are composed to make larger units and decomposed to make smaller units. Measurement itself requires viewing the attribute to be measured as composed of units. In effect, using a unit of measure to partition a continuous quantity, such as a length or area, into discrete and equal size pieces transforms it into a countable quantity.

Relating and Ordering

Relating and ordering allow one to decide which is more and which is less in various domains: number, length, area. Having children see and discuss relating and ordering across domains can deepen mathematical understanding. By broadening the ways in which things can be compared, children are led to the idea of different measurable attributes. For example, two stacks of blocks might be made from the same number of blocks, but one stack might be taller than the other. Relating is a first step toward measurement, because measurement is a quantified form of relating. A measurement specifies how many of one thing (the unit) it takes to make the other thing (the attribute that is measured). When relating and number are joined via measurement, both realms are extended. On one hand, relating becomes more precise when it becomes measurement, and, on the other hand, numbers extend into fractions and decimals in the context of measurement. For example, a bucket of sand might be filled with 2½ smaller pails of sand.

Looking for Patterns and Structures and Organizing Information

Looking for patterns and structures and organizing information (including classifying) are crucial mathematical processes used frequently in mathematical thinking and problem solving. They also have been viewed as distinct content areas in early childhood mathematics learning. Such pattern "content" usually focuses on repeated patterns, such as *abab* or *abcabc*, that are done with colors, sounds, body movements, and so forth (such as the bead and block patterning examples discussed in the section on unitizing). Such activities are appropriate in early childhood and can help to introduce children to seeing and describing patterns more broadly in mathematics. The patterns *abab*, *abcabc*, and *aabbaabb* can be learned by many young children, and many children in kindergarten can do more complex patterns (Clements and Sarama, 2007). Learning to see the unit in one direction (from left to right or from top to bottom or bottom to top) (*ab* in *abab*, *abc* in *abcabc*) and then repeating it consistently is the core of such repeated pattern learning. Learning to extend a given pattern to other modalities (for example, from color to shape, sounds, and body movements) is an index of abstracting and generalizing the pattern.

Counting involves some especially important patterns that go beyond simple repeating patterns. For example, the pattern of counting is a critical idea in number. The list of counting numbers has an especially important and intricate pattern, which involves a coordinated cycling of the digits 0 through 9 in the ones, tens, hundreds, etc., places (see Box 2-2). Although this intricate pattern will not be fully understood by children until later in elementary school, the foundation for this understanding is laid in early childhood as they identify and use the repeating patterns in the number words to 100.

Organizing information, including classifying, has also been seen as early childhood mathematics content, as children use attribute blocks and other collections of entities in which attributes are systematically varied so that they can sort them in multiple ways. Attribute blocks usually vary in color, shape, size, and sometimes thickness, so that children can sort on any of these dimensions and also describe a given block using multiple terms. For example, in small groups, a teacher may first ask children to sort the blocks on one or two dimensions: "Find all the big blue blocks." As children become more proficient, the teacher adds challenge, such as "Find the small blue thin rectangle." Later on, in preschool and in kindergarten, the teacher may ask children to generate their own descriptions of how groups of blocks are similar and different.

Recognizing patterns and organizing information are part of recognizing structure. At all levels in mathematics, one looks for structure. Some experiences in recognizing structure can be part of a foundation for later algebraic thinking. For example, recognizing that if there were 3 birds and then 2 more birds flew in versus if there were 2 birds at first and then 3 more birds flew in results in the same total number of birds either way is a step toward recognizing the commutative property of addition, that $a + b = b + a$ for all numbers a and b.

Although these content examples of looking for patterns and structures and organizing information are appropriate activities, they form a small part of the mathematics content for early childhood. Similarly, the specific skills in these examples are but a small part of the role that these processes play in mathematics.

MATHEMATICAL CONNECTIONS

In this section we discuss some of the main connections across content areas of early childhood mathematics and into later mathematics. Mathematics as a whole is a web of interconnected ideas, and the mathematics of early childhood is no exception. Mathematics is also deep, in that every mathematical idea, including those of early childhood, is embedded in long chains of related ideas. As this section shows, the foundational and achievable mathematical ideas discussed in the previous sections are tightly interwoven with each other and with other important ideas that are studied later in mathematics.

Connections in Structuring Numbers, Shapes, and Space

Throughout mathematics, structure is found and analyzed by composing and decomposing. A group of objects can be joined to form a new composite object. An object can be decomposed to reveal its finer structure. Some of the most important connections in elementary mathematics concern structuring of numbers and space via composition and decomposition. We now discuss several of these connections.

Making Units by Grouping

Numbers are structured by composition because the decimal place-value system relies on grouping by tens. In the realm of number, 10 individual counters are viewed as forming a single composite unit of 10. A geometric version of this grouping idea occurs when several shapes are put together to form another larger shape, which is then viewed as a unified shape in its own right, such as if the unified shape is seen as a possible substitute for another shape or as able to fill a space in a puzzle.

When children (or adults) make a repeating pattern, they might focus mainly on maintaining a certain order. But repeating patterns can also be viewed as made from a single composite unit that is copied over and over. This is not unlike viewing the counting numbers as a sequence that is structured in groups of 10 (see Figure 2-8).

Repeating patterns and, more generally, making groups of equal size are the basis for multiplication and division. Later in elementary school, when children skip count by fives, by counting 5, 10, 15, 20, . . . to list the multiples of 5, this pattern can be viewed as a growing pattern, but it can also be viewed as counting every fifth entry in a repeating pattern of 5. When children study division with remainders (in around fourth or fifth grade), they may observe a repeating pattern in the remainders. For example, when dividing successive counting numbers by 5, say, the remainders cycle through 0, 1, 2, 3, and 4.

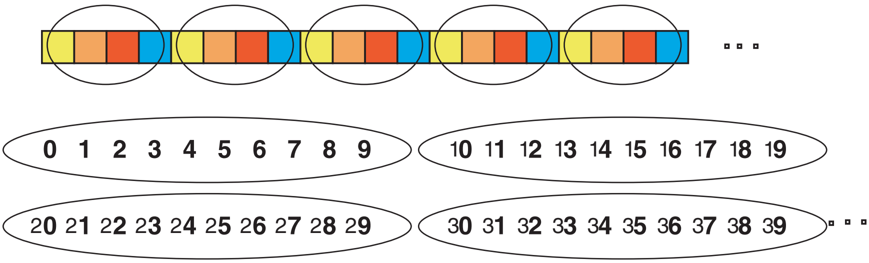

FIGURE 2-8 A repeating pattern is formed by repeating a unit. In counting, the ones digits form a repeating pattern.

Groups of Groups: Numbers, Shapes, and 2-D Space

The compositional structure of the decimal system is more complex than just making groups of 10 from 10 ones, since every 10 groups of 10 are composed into a unit of 100. A geometric version of this group's idea occurs when shapes are put together to form a new, composite shape, and composite shapes are then put together to make another composite shape—a composite of the composite shapes.

An especially important case of geometric structuring as composites of composites occurs when analyzing rectangles and their areas. When considering the area of a rectangle, one views the rectangle as composed of identical square tiles that cover the rectangle without gaps or overlaps. Each square tile has area one square unit. The area of the rectangle (in square units) is the number of squares that cover the rectangle. Although these squares can be counted one by one, to develop and understand the *length* × *width* formula for the area of a rectangle, the squares must be seen as grouped, either into rows or into columns (see Figure 2-6). Each row has the same number of squares in it, and the number of rows in the rectangle is equal to the number of squares in a column (likewise, each column has the same number of squares in it, and the number of columns is the number of squares in a row). Because of this grouping structure, the area of the rectangle is *# rows* × *# in each row* or *length* × *width* (square units). Similarly, the decimal system has a multiplicative structure because 100 is formed (by definition) by making 10 groups of 10, and so 100 = 10 × 10.

The idea of structuring rectangles as arrays of squares can be extended to structuring an entire infinite plane (in the imagination) as an infinite array of squares. This idea of a plane structured by an infinite array is essentially the idea of the Cartesian coordinate plane, in which each point in the plane is described by a pair of numbers that indicate its location relative to two coordinate lines (axes) (see Figure 2-9).

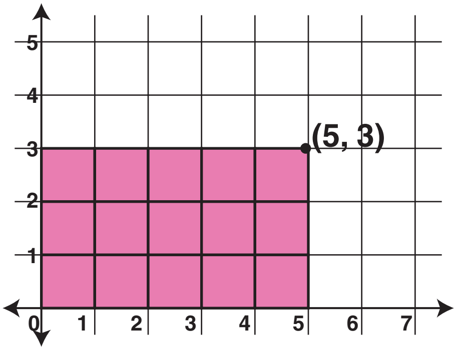

FIGURE 2-9 The coordinate plane.

Groups of Groups of Groups: Numbers, Shapes, and 3-D Space

The compositional structure of the decimal system consists not only of making groups of 10 from 10 ones and groups of 100 from 10 groups of 10, but also groups of 1,000 from 10 groups of 100, so that 1,000 = 10 × 10 × 10. The grouping structure of the decimal system continues in such a way that all successive groupings are obtained by repeatedly grouping by 10. The geometric counterpart of this grouping structure of the decimal system takes one into 3-D space and then higher dimensional space. Just as 2-D rectangles can be structured as 2-D arrays of squares, so, too, 3-D rectangular prisms (box shapes) can be structured as 3-D arrays of cubes. As in the case of rectangles, the multiplicative structure of a 3-D array of cubes explains why one multiplies the three dimensions of length, width, and height of a box to find its volume. Box shapes can be built as layers of identical cubes, as in Figure 2-12, and each layer can be viewed as groups of rows, so a box built from cubes can be viewed as a group of a group of cubes in the same way that 1,000 is 10 groups of 10 groups of 10.

When one extends the array structure of rectangular prisms to all of 3-D space, one gets essentially the idea of coordinate space, in which the location of each point in space is described by a triple of numbers that indicate its location relative to three coordinate lines.

Motion, Decomposing and Composing, Symmetry, and Properties of Arithmetic

The properties (or laws) of arithmetic are the fundamental structural properties of addition and multiplication from which all of arithmetic is derived. These properties include the commutative properties of addition and of multiplication, the associative properties of addition and multiplication, and the distributive property of multiplication over addition. The commutative properties of addition and multiplication state that

$$A + B = B + A \text{ for all numbers A, B}$$

$$A \times B = B \times A \text{ for all numbers A, B.}$$

The associative properties of addition and multiplication state that

$$A + (B + C) = (A + B) + C \text{ for all numbers A, B, C}$$

$$A \times (B \times C) = (A \times B) \times C \text{ for all numbers A, B, C.}$$

The distributive property states that

$$A \times (B + C) = A \times B + A \times C \text{ for all numbers A, B, C.}$$

Each property can be illustrated by moving and reorganizing objects, sometimes also by decomposing and recomposing a grouping, and sometimes even in terms of symmetry.

The report *Adding It Up: Helping Children Learn Mathematics* has a good discussion and an illustration of the commutative and associative properties of addition, the commutative and associative properties of multiplication, and the distributive property (National Research Council, 2001, Chapter 3 and Box 3-1). The commutative property of addition is illustrated by switching the order in which two sets are shown. The commutative property is especially useful in conjunction with counting on strategies for solving addition problems (see Chapter 5 for further discussion of children's problem-solving strategies for addition and subtraction). For example, instead of counting on 6 from 2 to calculate 2 + 6, a child can switch the problem to 6 + 2 and count on 2 from 6. The associative property involves starting with three separate sets, two of which are close together, separating the two that are close together, and moving one of those sets to reassociate with the other set. The associative property of addition is used in make-a-ten methods, when one number is decomposed so that one of the pieces can be recomposed with another number to make a group of 10.

Early experiences with properties of addition then extend to multiplication in third and fourth grade. The commutative and associative properties of multiplication and the distributive property are essential to understanding relationships among basic multiplication facts and to understanding multidigit multiplication and division. For example, knowing that $3 \times 5 = 5 \times 3$ and that 7×8 can be obtained by adding 5×8 and 2×8 lightens the load in learning the multiplication tables. The commutative property of multiplication is illustrated by decomposing a rectangular array in two different ways: by the rows or by the columns (as shown in Figure 2-6) or in terms of a rotation (see National Research Council, 2001, Box 3-1). The associative property of multiplication can be illustrated by decomposing a 3-D array (or box shape built of blocks) in different ways (one way is shown in Figure 2-7). The distributive property is illustrated by viewing objects as grouped in two different ways (see National Research Council, 2001, Box 3-1).

The properties of multiplication can be illustrated with arrays and rectangles, and they are also visible in the multiplication tables, which contain many relationships and have important structure. One structural aspect of the multiplication tables is their diagonal symmetry. This diagonal symmetry corresponds with the commutative property of multiplication, namely that $a \times b = b \times a$ for all numbers a and b. Recognizing this symmetry allows children to learn multiplication facts more efficiently. In other words, once they know the upper right-hand triangular portion of the multiplication tables in around third grade, they can fill in the rest of the table by reflecting across the diagonal (see Figure 2-10).

Patterns associated with horizontal or vertical shifts (slides) can also be seen in the multiplication tables. For example, the entries in two columns are related by the column that is associated with the amount of shift between the columns (see Figure 2-10). This structural relationship corresponds with the distributive property.

Connections in Measurement and Number: Fractions

Once children encounter measurement situations, the possibility of fractions arises naturally. Fractions can be shown well in the context of

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

The diagonal symmetry in the multiplication table is due to the commutative property.

shifting over by 2 adds the 2× column

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

The relationship among the columns is due to the distributive property, e.g.,
$6 \times 7 = 6 \times 5 + 6 \times 2$.

FIGURE 2-10 Symmetry and relationships in the multiplication table.

length and on number lines (in around second or third grade). A number line is much like an infinitely long ruler, so number lines can be viewed as unifying measurement and number in a one-dimensional space. A number on a number line can be thought of as representing the length from 0 to the number (see Figure 2-11).

Because of the close connection between number lines and length, number lines are difficult for children below about second grade. In contrast, the number paths on most number board games used for preschoolers are a count model, not a number line. There is a path of squares, circles, or rocks, each has a number on it, and players move along this path by counting the squares or other objects or saying the number on them as they move. These are appropriate for younger children because they can support their knowledge of counting, cardinality, comparing, and number symbols.

In measurement, there is an important relationship between the size of a unit and the number of units it takes to make a given, fixed quantity. For example, if the triangle in Figure 2-5 is designated to have 1 unit of area, then the hexagon has an area of 6 units. But if one picks a new unit of area, such as designating the area of the rhombus in Figure 2-5 to be 1 unit, which is twice the size of the triangle, then the hexagon has an area of only 3 units.

Later in elementary school (in around second grade), children see this inverse relationship between the size of a unit of measurement and the number of units it takes to make a given quantity reflected in the inverse relationship between the ordering of the counting numbers and the ordering of the unit fractions (see Figure 2-12).

Connections in Data Analysis, Number, and Measurement

To use data to answer (or address) a question, one must analyze the data, which often involves classifying the data into different categories,

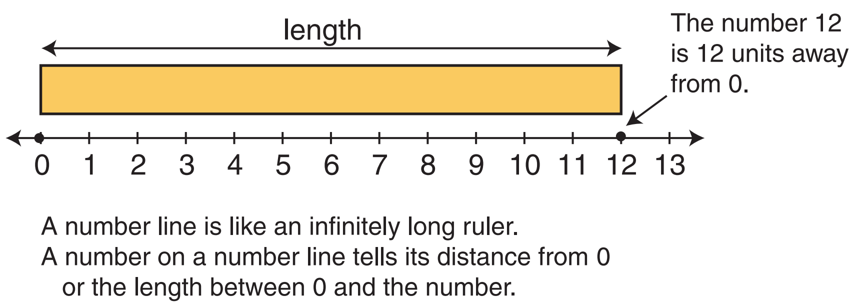

FIGURE 2-11 Number lines relate numbers to lengths.

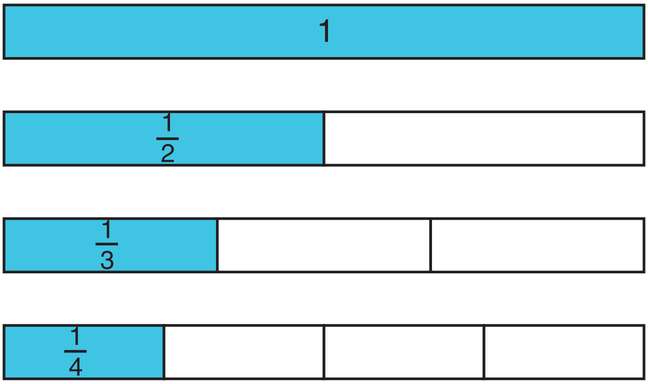

FIGURE 2-12 1 > ½ > ⅓ > ¼.

displaying the categorized data graphically, and describing or comparing the categories. Because the process of describing or comparing categories usually involves number or measurement, number and measurement are central to data analysis, and data analysis provides a context to which number and measurement can be applied.

The collection of data should ideally start with a question of interest to children. For example, children in a class might be interested in how everyone got to school in the morning and might wonder what way was most popular. To answer this question, children might divide themselves into different groups according to how they got to school in the morning (by bus, by car, by walking, or by bike). The children could then make "real graphs" (graphs made of real objects) either by lining up in their categories or by each placing a small toy or token to represent a bus, a car, a pair of shoes, or a bike into predrawn squares, as shown on the left in Figure 2-13 (the predrawn squares ensure that each object occupies the same amount of space in the graph). Instead of a real graph, children could display the data somewhat more abstractly in a pictograph by lining up sticky notes in categories, as on the right in the figure. Each child places a sticky note in the category for how the child got to school.

In general, pictographs use small, identical pictures to represent data. In this case, each sticky note stands for a single piece of data and functions as a small picture in a pictograph. Children can then use these real graphs or pictographs to answer such questions as "How many children rode a bus to get to school today?" or "Did more children ride in a car or walk to school today?" or even "If it were raining today, how do you think the graph might be different?" Data displays that are used in posing and answering such quantitative questions serve a purpose and help children mathematize their daily experiences. In contrast, data displays that are only

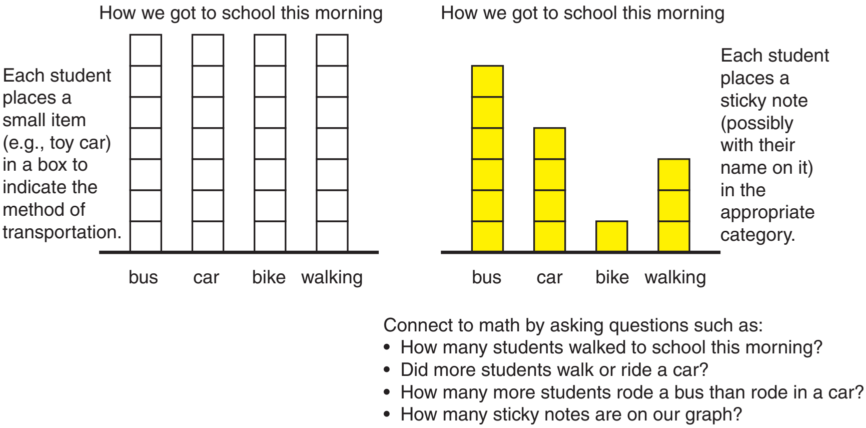

FIGURE 2-13 A template for a "real graph" and a pictograph made with sticky notes.

made but not discussed are not likely to help children develop or extend their mathematical thinking.

In around second or third grade, once children have worked with linear measurement, they can begin to work with bar graphs. One can think of bar graphs as arising from pictographs by fusing the separated entries in a pictograph to make the bars in a bar graph. In this way, the discrete counting of separate entries in a pictograph gives way to the length measurement of a bar in a bar graph.

In third grade or so, once children have begun to skip count and to multiply, the entries in a pictograph can be used to represent more than one single piece of data. For example, each picture might represent 2 pieces of data or 10 pieces of data.

SUMMARY

This chapter describes the foundational and achievable mathematics content for young children. The focus of this chapter is on the mathematical ideas themselves rather than on the teaching or learning of these ideas. These mathematical ideas are often taken for granted by adults, but they are surprisingly deep and complex. There are two fundamental areas of mathematics for young children: (1) number and (2) geometry and measurement as identified in NCTM's Curriculum Focal Points and outlined by this committee. There are also important mathematical reasoning processes that children must engage in. This chapter also describes some of the most important connections of the mathematics for young children to later mathematics.

In the area of number, a fundamental idea is the connection between the counting numbers as a list and for describing how many objects are in a set. We can represent arbitrarily large counting numbers in an efficient, systematic way by means of the remarkable decimal system (base 10). We can use numbers to compare quantities without matching the quantities directly. The operations of addition and subtraction allow us to describe how amounts are related before and after combining or taking away, how parts and totals are related, and to say precisely how two amounts compare.

In the area of geometry and measurement, a fundamental idea is that geometric shapes have different parts and aspects that can be described, and they can be composed and decomposed. To measure the size of something, one first selects a specific measurable attribute of the thing, and then views the thing as composed of some number of units. The shapes of geometry can be viewed as idealized and simplified approximations of objects in the world. Space has structure that derives from movement through space and from relative location within space. An important way to think about the structure of 2-D and 3-D space comes from viewing rectangles as composed of rows and columns of squares and viewing box shapes as composed of layers of rows and columns of cubes.

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